**Homework 3, due 9/16**

1. Suppose that $a, b \in \mathbf{C}$, and $|a| < r < |b|$. Show that

$$\int_\gamma \frac{dz}{(z-a)(z-b)} = \frac{2\pi i}{a-b},$$

where $\gamma(t) = re^{2\pi i t}$, $t \in [0,1]$.

2. Suppose that $f : \mathbf{C} \to \mathbf{C}$ is holomorphic, and there are constants $A, B, n > 0$ such that $|f(z)| \leq A|z|^n + B$ for all $z \in \mathbf{C}$. Prove that $f$ is a polynomial.

3. Prove that if $N > 0$ is an integer and $f$ is holomorphic on $D(0,2)$ with

$$|f^{(N)}(0)| = N! \sup\{|f(z)| \,:\, |z| = 1\},$$

then $f(z) = cz^N$ for some $c \in \mathbf{C}$.

4. Suppose that $f, g : \mathbf{C} \to \mathbf{C}$ are holomorphic, and $|f(z)| \leq |g(z)|$ for all $z \in \mathbf{C}$. Prove that $f = cg$ for some $c \in \mathbf{C}$.

5. Suppose that $f$ is holomorphic on the disk $D(0,2)$. In this problem we will give two different proofs of Cauchy's inequality in the form that there is a constant $C > 0$ such that

$$|f'(0)| \leq C \sup\{|f(z)| \,:\, |z| = 1\}.$$

   (a) Prove the inequality using the maximum principle, by showing that for a suitable cutoff function $\eta : D(0,1) \to \mathbf{R}$ vanishing on the boundary, and a suitable constant $D > 0$ we have

$$\Delta(\eta^2 |f'|^2 + D|f|^2) \geq 0.$$

   (b) Prove the inequality using Liouville's theorem and an argument by contradiction: if no suitable $C$ were to exist, then we would have a sequence of holomorphic functions $f_k$ on $D(0,2)$ with $\sup\{|f_k(z)| : |z| = 1\} \leq 1$ and $|f_k'(0)| = k$. Use these $f_k$ to construct a bounded, non-constant holomorphic function on $\mathbf{C}$.

6. Suppose that $f_n : \mathbf{C} \to \mathbf{C}$ are holomorphic functions with only real zeros, and that $f_n \to f$ locally uniformly on $\mathbf{C}$. Show that $f$ has only real zeros, unless $f$ is identically zero.

7. Suppose that $f(z) = \sum_{n \geq 0} c_n z^n$ defines a holomorphic function on $D(0,1)$, such that $f(z) \in \mathbf{R}$ for all $z \in D(0,1) \cap \mathbf{R}$. Show that $c_n \in \mathbf{R}$ for all $n$.